On Strongly Regular Graphs, Friendship, and the Shannon Capacity

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Graph Spectrum

Throughout this presentation,

- $\mathsf{G} = (\mathsf{V}(\mathsf{G}), \mathsf{E}(\mathsf{G}))$ is a finite, undirected, and simple graph of order $|\mathsf{V}(\mathsf{G})| = n$ and size $|\mathsf{E}(\mathsf{G})| = m$.
- $\mathbf{A} = \mathbf{A}(\mathsf{G})$ is the *adjacency matrix* of the graph.
- The eigenvalues of $\mathbf{A}$ are given in decreasing order by

$$\lambda_{\max}(\mathsf{G}) = \lambda_1(\mathsf{G}) \geq \lambda_2(\mathsf{G}) \geq \ldots \geq \lambda_n(\mathsf{G}) = \lambda_{\min}(\mathsf{G}). \tag{1.1}$$

- The *spectrum* of G is a multiset that consists of all the eigenvalues of $\mathbf{A}$, including their multiplicities.

Orthogonal Representation of Graphs

Definition 1.1

Let G be a finite, undirected and simple graph.
An orthogonal representation of G in $\mathbb{R}^d$

$$i \in \mathsf{V}(\mathsf{G}) \; \mapsto \; \mathbf{u}_i \in \mathbb{R}^d$$

such that

$$\mathbf{u}_i^{\mathrm{T}} \mathbf{u}_j = 0, \quad \forall \, \{i, j\} \notin \mathsf{E}(\mathsf{G}).$$

An orthonormal representation of G: $\|\mathbf{u}_i\| = 1$ for all $i \in \mathsf{V}(\mathsf{G})$.

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In an orthogonal representation of a graph G:

- non-adjacent vertices: mapped to orthogonal vectors;
- adjacent vertices: not necessarily mapped to non-orthogonal vectors.

Lovász ϑ-function

Let G be a finite, undirected and simple graph.

The Lovász ϑ-function of G is defined as

$$\vartheta(\mathsf{G}) \triangleq \min_{\mathbf{u},\mathbf{c}} \max_{i \in \mathsf{V}(\mathsf{G})} \frac{1}{\left(\mathbf{c}^{\mathrm{T}} \mathbf{u}_i\right)^2}, \tag{1.2}$$

where the minimum is taken over

- all orthonormal representations $\{\mathbf{u}_i : i \in \mathsf{V}(\mathsf{G})\}$ of G, and
- all unit vectors $\mathbf{c}$.

The unit vector $\mathbf{c}$ is called the *handle* of the orthonormal representation.

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$$\left|\mathbf{c}^{\mathrm{T}} \mathbf{u}_i\right| \le \|\mathbf{c}\| \, \|\mathbf{u}_i\| = 1 \implies \vartheta(\mathsf{G}) \ge 1,$$

with equality if and only if G is a complete graph.

An Orthonormal Representation of a Pentagon

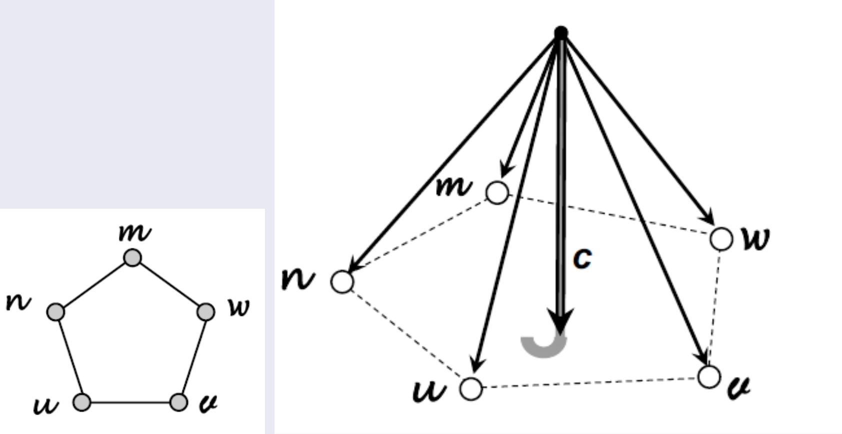

Figure 1: A 5-cycle graph and its orthonormal representation (also known as Lovász umbrella). Calculation shows that $\vartheta(\mathrm{C}_5) = \sqrt{5}$ (Lovász, 1979).

Lovász ϑ-function (Cont.)

- $\mathbf{A}$ is the $n \times n$ adjacency matrix of G $\quad (n \triangleq |\mathsf{V}(\mathsf{G})|)$;
- $\mathbf{J}_n$ is the all-ones $n \times n$ matrix;
- $\mathcal{S}_+^n$ is the set of all $n \times n$ positive semidefinite matrices.

Semidefinite program (SDP), with strong duality, for computing $\vartheta(\mathsf{G})$:

$$\begin{array}{l} \text{maximize} \;\; \mathrm{Trace}(\mathbf{B}\,\mathbf{J}_n) \\ \text{subject to} \\ \begin{cases} \mathbf{B} \in \mathcal{S}_+^n, \;\; \mathrm{Trace}(\mathbf{B}) = 1, \\ A_{i,j} = 1 \;\Rightarrow\; B_{i,j} = 0, \quad i, j \in [n]. \end{cases} \end{array}$$

Computational complexity: $\exists$ algorithm (based on the ellipsoid method) that numerically computes $\vartheta(\mathsf{G})$, for every graph G, with precision of r decimal digits, and polynomial-time in n and r.

Lovász ϑ-function (Cont.)

Let $\alpha(G)$, $\omega(G)$, and $\chi(G)$ denote the independence number, clique number, and chromatic number of a graph G. Then,

1. Sandwich theorem:
$$\alpha(G) \leq \vartheta(G) \leq \chi(\overline{G}), \tag{1.3}$$
$$\omega(G) \leq \vartheta(\overline{G}) \leq \chi(G). \tag{1.4}$$

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2. Computational complexity:
   - $\alpha(\mathsf{G})$, $\omega(\mathsf{G})$, and $\chi(\mathsf{G})$ are NP-hard problems.
   - However, the numerical computation of $\vartheta(\mathsf{G})$ is in general feasible by convex optimization (SDP problem).

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3. Hoffman-Lovász inequality: Let G be d-regular of order n. Then,
$$\vartheta(\mathsf{G}) \leq -\frac{n\,\lambda_n(\mathsf{G})}{d - \lambda_n(\mathsf{G})}, \tag{1.5}$$
with equality if G is edge-transitive.

Strongly Regular Graphs

Let G be a d-regular graph of order n. It is a *strongly regular* graph (SRG) if there exist nonnegative integers λ and μ such that

- Every pair of adjacent vertices have exactly λ common neighbors;
- Every pair of distinct and non-adjacent vertices have exactly μ common neighbors.

Such a strongly regular graph is denoted by $\text{srg}(n, d, \lambda, \mu)$.

Theorem 1.2 (Bounds on Lovász function of Regular Graphs, I.S., '23)

Let G be a d-regular graph of order n, which is a non-complete and non-empty graph. Then, the following bounds hold for the Lovász ϑ-function of G and its complement $\overline{\mathsf{G}}$:

1)

$$\frac{n - d + \lambda_2(\mathsf{G})}{1 + \lambda_2(\mathsf{G})} \leq \vartheta(\mathsf{G}) \leq -\frac{n\lambda_n(\mathsf{G})}{d - \lambda_n(\mathsf{G})}. \tag{1.6}$$

- Equality holds in the leftmost inequality if $\overline{\mathsf{G}}$ is both vertex-transitive and edge-transitive, or if G is a strongly regular graph;
- Equality holds in the rightmost inequality if G is edge-transitive, or if G is a strongly regular graph.

Cont. of Theorem 1.2

2)

$$1 - \frac{d}{\lambda_n(\mathsf{G})} \leq \vartheta(\overline{\mathsf{G}}) \leq \frac{n\big(1+\lambda_2(\mathsf{G})\big)}{n-d+\lambda_2(\mathsf{G})}. \tag{1.7}$$

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A Common Sufficient Condition

All inequalities hold with equality if G is strongly regular. (Recall that the graph G is strongly regular if and only if $\overline{\mathsf{G}}$ is so).

Lovász Function of Strongly Regular Graphs (I.S., '23)

Let G be a strongly regular graph with parameters srg(n, d, λ, μ). Then,

$$\vartheta(\mathsf{G}) = \frac{n\,(t+\mu-\lambda)}{2d+t+\mu-\lambda}, \tag{1.8}$$

$$\vartheta(\overline{\mathsf{G}}) = 1 + \frac{2d}{t+\mu-\lambda}, \tag{1.9}$$

where

$$t \triangleq \sqrt{(\mu-\lambda)^2 + 4(d-\mu)}. \tag{1.10}$$

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New Relation for Strongly Regular Graphs

$$\vartheta(\mathsf{G})\,\vartheta(\overline{\mathsf{G}}) = n, \tag{1.11}$$

holding not only for all vertex-transitive graphs (Lovász '79), but also for all strongly regular graphs (that are not necessarily vertex-transitive).

We next provide an original proof of the following celebrated theorem by Erdös, Rényi and Sós (1966), based on our expression for the Lovász ϑ-function of strongly regular graphs (and their complements, which are also strongly regular graphs).

Theorem 1.3 (Friendship Theorem)

Let G be a finite graph in which any two distinct vertices have a single common neighbor. Then, G has a vertex that is adjacent to every other vertex.

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Let G be a finite graph in which any two distinct vertices have a single common neighbor. Then, G has a vertex that is adjacent to every other vertex.

A Human Interpretation of Theorem 1.3

- There is a party with n people, where every two people have precisely one common friend in that party.
- Theorem 1.3 asserts that one of these people is everybody's friend.
- Indeed, construct a graph whose vertices represent the n people, and every two vertices are adjacent if and only if they represent two friends. The claim then follows from Theorem 1.3.

Remark 1 (On the Friendship Theorem - Theorem 1.3)

- The windmill graph (see Figure 2) has the desired property, and it turns out to be the only one graph with that property.
- Remarkably, the friendship theorem does not hold for infinite graphs. Indeed, for an inductive construction of a counterexample, one may start with a 5-cycle C_5, and repeatedly add a common neighbor for every pair of vertices that does not yet have one. This process results in a countably infinite friendship graph without a vertex adjacent to all other vertices.

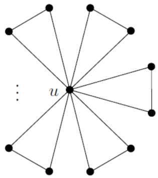

Figure 2: Windmill graph.

Alternative Proof of Theorem 1.3 (Cont.)

Suppose the assertion is false, and G is a counterexample. In other words, there exists one vertex in G that is not adjacent to all other vertices.

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The first step shows that the graph G is regular, as proved by Aigner and Ziegler, *Proofs from THE BOOK, 6th Edition, Chapter 44*. We provide a variation of that proof, and then the rest of our proof proceeds differently

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To assert the regularity of G, it is first proved that nonadjacent vertices in G have equal degrees, i.e., $d(u) = d(v)$ if $\{u, v\} \notin \mathsf{E}(\mathsf{G})$.

- The given hypothesis yields that G is a connected graph.

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- Let $\{u, v\} \notin \mathrm{E}(\mathrm{G})$, and let $\mathcal{N}(u)$ and $\mathcal{N}(v)$ denote, respectively, the sets of neighbors of the nonadjacent vertices u and v.
- Let $f \colon \mathcal{N}(u) \to \mathcal{N}(v)$ be the injective function where every $x \in \mathcal{N}(u)$ is mapped to the unique $y \in \mathcal{N}(x) \cap \mathcal{N}(v)$. Indeed, if $z \in \mathcal{N}(u) \setminus \{x\}$ satisfies $f(z) = y$, then x and z share two common neighbors (namely, y and u), which contradicts the assumption of the theorem.

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- By symmetry, swapping u and v (as nonadjacent vertices) also yields $|\mathcal{N}(v)| \leq |\mathcal{N}(v)|$, so $d(u) = |\mathcal{N}(u)| = |\mathcal{N}(v)| = d(v)$ for all vertices $u, v \in \mathsf{V}(\mathsf{G})$ such that $\{u, v\} \notin \mathsf{E}(\mathsf{G})$.

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- To complete the proof that G is regular, let u and v be nonadjacent vertices in G. By assumption, except of one vertex, all vertices are either nonadjacent to u or v. Hence, except of that vertex, all these vertices must have identical degrees by what we already proved.

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- Finally, by our further assumption (later leading to a contradiction), since there is no vertex in G that is adjacent to all other vertices, also the single vertex that is adjacent to u and v has a nonneighbor in G, so it also should have an identical degree to all the degrees of the other vertices by what is proved above.

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- Consequently, G is a regular graph.

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- First, if $k = 1$ or $k = 2$, then by assumption, it follows that $\mathsf{G} = \mathsf{K}_1$ or $\mathsf{G} = \mathsf{K}_2$, respectively, leading to a contradiction. Hence, let $k \geq 3$.

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- Every two adjacent vertices in G share a common neighbor, so G contains a triangle. Moreover, G is C_4-free since every two vertices have exactly one common neighbor, so it is K_4-free. Hence, $\omega(\mathsf{G}) = 3$.

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- We next show that $\chi(\mathrm{G}) = 3$. First, $\chi(\mathrm{G}) \geq \omega(\mathrm{G}) = 3$. We also need to show that $\chi(\mathrm{G}) \leq 3$, which means that three colors suffice to color the vertices of G in a way that no two adjacent vertices are assigned the same color. This can be done recursively by noticing that every edge belongs to exactly one triangle, and a newly colored vertex always complete a properly colored triangle, ensuring that at each step, the coloring remains valid without requiring a fourth color.

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- By the sandwich theorem $\omega(\mathsf{G}) \leq \vartheta(\overline{\mathsf{G}}) \leq \chi(\mathsf{G})$, so $\vartheta(\overline{\mathsf{G}}) = 3$.

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- This leads to a contradiction since, for all $k \geq 3$,

$$\begin{aligned} &(k-2)^2 > 0, \\ \Leftrightarrow\ & k^2 > 4(k-1), \\ \Leftrightarrow\ & 1 + \frac{k}{\sqrt{k-1}} > 3. \end{aligned}$$

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I. Sason, "On strongly regular graphs and the friendship theorem," submitted, February 2025. `https://arxiv.org/abs/2502.13596`

A Second Alternative Proof of Theorem 1.3

From the point where we get, by contradiction, that G is srg$(n, k, 1, 1)$, it is possible to get a contradiction in the following alternative way.

Proposition 1.1 (Feasible Parameters of Strongly Regular Graphs)

Let G be a strongly regular graph with parameters srg(n, d, λ, μ). Then,

1. $(n - d - 1)\,\mu = d\,(d - \lambda - 1)$.
2. $\frac{2d+(n-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(d-\mu)}}$ is an integer whose absolute value is less than $n - 1$.
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Proof

- Condition 1 is a combinatorial equality for strongly regular graphs.
- Condition 2 holds by the integrality of the multiplicities of the second-largest and least eigenvalues of the adjacency matrix.
- Condition 3 holds by the number of triangles in the graph G.

A Second Alternative Proof of Theorem 1.3 (Cont.)

- By Item 1 in Proposition 1.1 with $d = k$ and $\lambda = \mu = 1$, we get $n = k^2 - k + 1$. This does not lead to a contradiction since summing over all the degrees of the neighbors of an arbitrary vertex u gives k^2. Then, by the assumption of the theorem that every two vertices have exactly one common neighbor, it follows that the above summation counts every vertex in G exactly one time, except of u that is counted k times. Hence, indeed $n = k^2 - k + 1$.

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- By Item 2 in Proposition 1.1 with $d = k$ and $\lambda = \mu = 1$, we get that $\frac{k}{\sqrt{k-1}} \in \mathbb{N}$. Consequently, $(k-1)|k^2 \in \mathbb{N}$. Since $k^2 = (k-1)(k+1) + 1$, it follows that $(k-1)|1$, so $k = 2$. If $k = 2$, the only graph that satisfies the condition of Theorem 1.3 is $\mathrm{G} = \mathrm{K}_2$, which also satisfies the assertion of the theorem. Hence, this argument contradicts the assumption in the proof since it led to the conclusion that G is srg$(n, k, 1, 1)$.

The sandwich theorem for the Lovász ϑ-function applied to strongly regular graphs gives the following result.

Corollary 1.4 (Bounds on Parameters of SRGs)

Let G be a strongly regular graph with parameters $\text{srg}(n, d, \lambda, \mu)$. Then,

$$\alpha(\mathsf{G}) \leq \left\lfloor \frac{n\,(t+\mu-\lambda)}{2d+t+\mu-\lambda} \right\rfloor \tag{1.12}$$

$$\omega(\mathsf{G}) \leq 1 + \left\lfloor \frac{2d}{t+\mu-\lambda} \right\rfloor, \tag{1.13}$$

$$\chi(\mathsf{G}) \geq 1 + \left\lceil \frac{2d}{t+\mu-\lambda} \right\rceil, \tag{1.14}$$

$$\chi(\overline{\mathsf{G}}) \geq \left\lceil \frac{n\,(t+\mu-\lambda)}{2d+t+\mu-\lambda} \right\rceil, \tag{1.15}$$

with

$$t \triangleq \sqrt{(\mu-\lambda)^2 + 4(d-\mu)}. \tag{1.16}$$

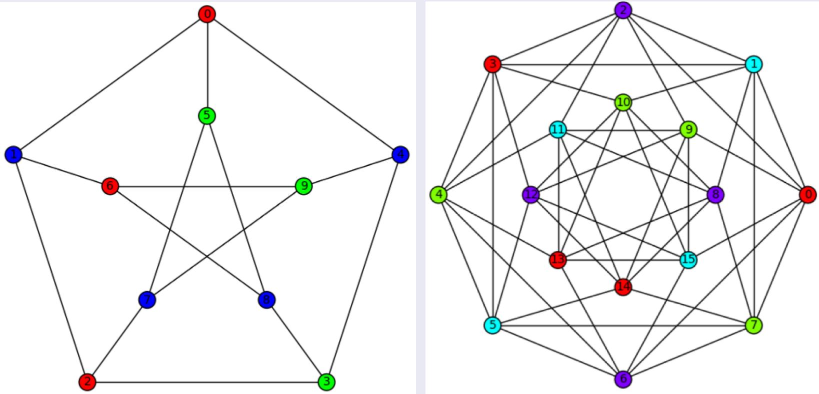

Figure 3: The Petersen graph is srg$(10, 3, 0, 1)$ (left), and the Shrikhande graph is srg$(16, 6, 2, 2)$ (right). Their chromatic numbers are 3 and 4, respectively.

Schläfli Graph

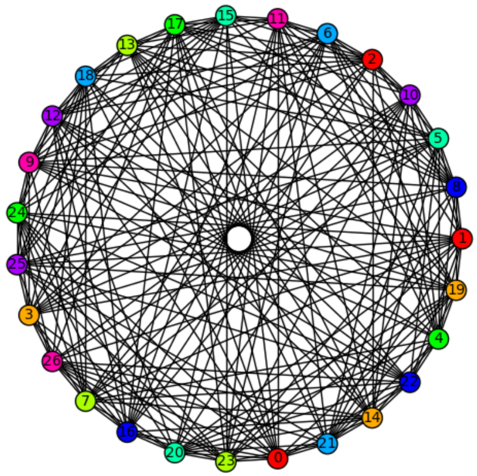

Figure 4: Schläfli graph is $\mathrm{srg}(27, 16, 10, 8)$ with chromatic number $\chi(\mathrm{G}) = 9$.

Examples: Bounds on Parameters of SRGs (Cont.)

1. Let G_1 be the Petersen graph. Then, the bounds on the independence, clique, and chromatic numbers of G are tight:

$$\alpha(\mathsf{G}_1) = 4, \quad \omega(\mathsf{G}_1) = 2, \quad \chi(\mathsf{G}_1) = 3. \tag{1.17}$$

2. The bounds on the chromatic numbers of the Schläfli graph (G_2), Shrikhande graph (G_3) and Hall-Janko graph (G_4) are tight:

$$\chi(\mathsf{G}_2) = 9, \quad \chi(\mathsf{G}_3) = 4, \quad \chi(\mathsf{G}_4) = 10. \tag{1.18}$$

3. For the Shrikhande graph (G_3),
   - the bound on its independence number is also tight: $\alpha(\mathsf{G}_3) = 4$,
   - its upper bound on its clique number is, however, not tight (it is equal to 4, and $\omega(\mathsf{G}_3) = 3$).

Strong Product of Graphs

Let G and H be two graphs. The strong product $\mathsf{G} \boxtimes \mathsf{H}$ is a graph with

- vertex set: $\mathsf{V}(\mathsf{G} \boxtimes \mathsf{H}) = \mathsf{V}(\mathsf{G}) \times \mathsf{V}(\mathsf{H})$,
- two distinct vertices (g, h) and (g', h') in $\mathsf{G} \boxtimes \mathsf{H}$ are adjacent if the following two conditions hold:
  1. $g = g'$ or $\{g, g'\} \in \mathsf{E}(\mathsf{G})$,
  2. $h = h'$ or $\{h, h'\} \in \mathsf{E}(\mathsf{H})$.

Strong products are commutative and associative.

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Strong products are commutative and associative.

Strong Powers of Graphs

Let

$$\mathsf{G}^{\boxtimes k} \triangleq \underbrace{\mathsf{G} \boxtimes \ldots \boxtimes \mathsf{G}}_{\mathsf{G} \text{ appears } k \text{ times}}, \quad k \in \mathbb{N} \tag{1.19}$$

denote the k-fold strong power of a graph G.

Shannon Capacity of a Graph (1956)

- The capacity of a graph G was introduced by Claude E. Shannon (1956) to represent the maximum information rate that can be obtained with zero-error communication.

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- A channel is represented by a confusion graph G, where the vertices of G represent the input symbols and two vertices are adjacent if the corresponding pair of input symbols can be confused by the channel decoder). The Shannon capacity of a graph G is given by

$$\Theta(\mathsf{G}) = \sup_{k \in \mathbb{N}} \sqrt[k]{\alpha(\mathsf{G}^{\boxtimes k})}$$
$$= \lim_{k \to \infty} \sqrt[k]{\alpha(\mathsf{G}^{\boxtimes k})}. \tag{2.1}$$

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$$\Theta(\mathsf{G}) = \sup_{k \in \mathbb{N}} \sqrt[k]{\alpha(\mathsf{G}^{\boxtimes k})} = \lim_{k \to \infty} \sqrt[k]{\alpha(\mathsf{G}^{\boxtimes k})}. \tag{2.1}$$

- The last equality holds by Fekete's Lemma since the sequence $\{\log \alpha(\mathsf{G}^{\boxtimes k})\}_{k=1}^{\infty}$ is super-additive, i.e.,

$$\alpha(\mathsf{G}^{\boxtimes (k_1+k_2)}) \geq \alpha(\mathsf{G}^{\boxtimes k_1})\, \alpha(\mathsf{G}^{\boxtimes k_2}). \tag{2.2}$$

On the Computability of the Shannon Capacity of Graphs

- The Shannon capacity of a graph can be rarely computed exactly. ☹
- However, the Lovász ϑ-function of a graph is a computable (and sometimes tight) upper bound on the Shannon capacity. ☺

Lovász Bound on the Shannon Capacity of Graphs (1979)

Theorem: For every finite, simple and undirected graph G,

$$\Theta(\mathsf{G}) \leq \vartheta(\mathsf{G}). \tag{2.3}$$

Capacity of Graphs

Proposition: Let G be a finite, undirected, and simple graph. If $\alpha(\mathsf{G}^{\boxtimes \ell}) = \vartheta(\mathsf{G})^{\ell}$ for some $\ell \in \mathbb{N}$, then

$$\Theta(\mathsf{G}) = \vartheta(\mathsf{G}), \quad \forall\, k \in \mathbb{N}. \tag{2.4}$$

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Corollary 1: If $\alpha(\mathsf{G}) = \vartheta(\mathsf{G})$, then for all $k \in \mathbb{N}$, the k-fold strong power of G satisfies

$$\alpha(\mathsf{G})^k = \alpha(\mathsf{G}^{\boxtimes k}) = \Theta(\mathsf{G}^{\boxtimes k}) = \vartheta(\mathsf{G}^{\boxtimes k}) = \vartheta(\mathsf{G})^k, \quad \forall\, k \in \mathbb{N}. \tag{2.5}$$

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By Corollary 1 and our closed expression for the Lovász ϑ-function of strongly regular graphs, the Shannon capacity of some strongly regular graphs can be determined.

Shannon Capacities of Some Strongly Regular Graphs

1. The Hall-Janko graph G is $\text{srg}(100, 36, 14, 12)$, and $\Theta(G) = 10$.
2. The Hoffman-Singleton graph G is $\text{srg}(50, 7, 0, 1)$, and $\Theta(G) = 15$.
3. The Janko-Kharaghani graphs of orders 936 and 1800 are $\text{srg}(936, 375, 150, 150)$ and $\text{srg}(1800, 1029, 588, 588)$, respectively. The capacity of both graphs is 36.
4. Janko-Kharaghani-Tonchev: $G = \text{srg}(324, 153, 72, 72), \Theta(G) = 18$.
5. The graphs introduced by Makhnev are $G = \text{srg}(64, 18, 2, 6)$ and $\overline{G} = \text{srg}(64, 45, 32, 30)$. Capacities: $\Theta(G) = 16$, and $\Theta(\overline{G}) = 4$.
6. The Mathon-Rosa graph G is $\text{srg}(280, 117, 44, 52)$, and $\Theta(G) = 28$.
7. The Schläfli graph G is $\text{srg}(27, 16, 10, 8)$, and $\Theta(G) = 3$.
8. The Shrikhande graph is $\text{srg}(16, 6, 2, 2)$; its capacity is $\Theta(G) = 4$.
9. The Sims-Gewirtz graph G is $\text{srg}(56, 10, 0, 2)$, and $\Theta(G) = 16$.
10. The graph G by Tonchev is $\text{srg}(220, 84, 38, 28)$, and $\Theta(G) = 10$.

In some cases, the Shannon capacity of a graph can be calculated exactly, and the Lovász ϑ-function is a tight bound. ☺

Theorem 2.1 (Self-complementary vertex-transitive graphs, Lovász 79)

Let G be an undirected and simple graph on n vertices.

1. If G is a vertex-transitive graph on n vertices, then

$$\alpha(\mathrm{G} \boxtimes \overline{\mathrm{G}}) = \Theta(\mathrm{G} \boxtimes \overline{\mathrm{G}}) = \vartheta(\mathrm{G} \boxtimes \overline{\mathrm{G}}) = n. \tag{2.6}$$

2. If G is a self-complementary and vertex-transitive graph on n vertices, then

$$\Theta(\mathrm{G}) = \sqrt{n} = \vartheta(\mathrm{G}). \tag{2.7}$$

Theorem 2.2 (Strengthened and Refined Ver. of Thm. 2.1 (I.S., '24))

Let G be an undirected and simple graph on n vertices.

1. If G is a vertex-transitive or strongly regular graph, then
$$\alpha(\mathsf{G} \boxtimes \overline{\mathsf{G}}) = \Theta(\mathsf{G} \boxtimes \overline{\mathsf{G}}) = \vartheta(\mathsf{G} \boxtimes \overline{\mathsf{G}}) = n. \tag{2.8}$$
2. If G is a conference graph, then $\vartheta(\mathsf{G}) = \sqrt{n}$.
3. If G is a self-complementary graph with $\alpha(\mathsf{G}) = k$, then
$$\sqrt{n} \le \Theta(\mathsf{G}) \le 16\, n^{\frac{k-1}{k+1}}. \tag{2.9}$$
4. If G is a self-complementary graph that is vertex-transitive or strongly regular, then
$$\Theta(\mathsf{G}) = \sqrt{n} = \vartheta(\mathsf{G}), \tag{2.10}$$
$$\sqrt{\alpha(\mathsf{G} \boxtimes \mathsf{G})} = \Theta(\mathsf{G}). \tag{2.11}$$

Hence, the minimum Shannon capacity among all self-complementary graphs of a fixed order n is achieved by those that are vertex-transitive or strongly regular, and this minimum is equal to $\sqrt{n}$.

Summary (I.S., '23)

- Upper and lower bounds on the Lovász-ϑ function of regular graphs.
- These spectral bound depend on the second-largest and smallest eigenvalues of the adjacency matrix.
- The upper bound is due to Lovász, followed by a new sufficient condition for its tightness, and the lower bound is new.
- These bounds are tight $\iff$ the graph is strongly regular (SRG).
- Useful in bounding graph invariants, including the Shannon capacity.

Summary (I.S., '24)

Our follow-up published work (AIMS-Mathematics, 2024) delves into three research directions, leveraging the Lovász ϑ-function of graphs.

- It provides new results on the Shannon capacity of graphs, including the determination of that capacity for two infinite subclasses of SRGs.
- For every even integer $n \geq 14$, it is constructively proven that there exist connected, irregular, cospectral, and nonisomorphic graphs on n vertices such that the following holds:
  - Cospectrality with respect to the adjacency, Laplacian, signless Laplacian, and normalized Laplacian matrices,
  - They share identical independence, clique, and chromatic numbers,
  - Their Lovász ϑ-functions are distinct.
- A query regarding the variant of the ϑ-function by Schrijver and the identical function by McEliece *et al.* (1978) is resolved.
- It is shown, by a counterexample, that the ϑ-function variant by Schrijver does not possess the property of the Lovász ϑ-function of forming an upper bound on the Shannon capacity of a graph.

Recent Journal Papers

This talk presents in part the following recent journal papers:

1. I. Sason, "Observations on the Lovász ϑ-function, graph capacity, eigenvalues, and strong products," *Entropy*, vol. 25, no. 1, paper 104, pp. 1–40, January 2023. https://doi.org/10.3390/e25010104
2. I. Sason, "Observations on graph invariants with the Lovász ϑ-function," *AIMS Mathematics*, vol. 9, pp. 15385–15468, April 2024. https://www.aimspress.com/article/doi/10.3934/math.2024747
3. I. Sason, "On strongly regular graphs and the friendship theorem," submitted, February 2025. https://arxiv.org/abs/2502.13596